

# On Transiso Graph

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## Abstract

In this note, we define a new graph  $\Gamma_d(G)$  on a finite group  $G$ , where  $d$  is a divisor of  $|G|$ . The vertices of  $\Gamma_d(G)$  are the subgroups of  $G$  of order  $d$  and two subgroups  $H_1$  and  $H_2$  of  $G$  are said to be adjacent if there exists  $S_i \in \mathcal{T}(G, H_i)$  ( $i = 1, 2$ ) such that  $S_1 \cong S_2$ . We shall discuss the completeness of  $\Gamma_d(G)$  for various groups like finite abelian groups, dihedral groups and some finite  $p$ -groups.

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## 1 Introduction

Let  $G$  be a finite group and  $H$  be a subgroup of  $G$ . A *right transversal*  $S$  of  $H$  in  $G$  is a subset of  $G$  obtained by selecting one and only one element from each right coset of  $H$  in  $G$ .  $S$  is *normalized right transversal (NRT)* if  $1 \in S$ . An NRT  $S$  has an induced binary operation  $\circ$  given by  $\{x \circ y\} = S \cap Hxy$ , with respect to which  $S$  is a right loop with identity 1, that is, a right quasigroup with both sided identity (see [17, Proposition 2.2, p.42], [14]). Conversely, every right loop can be embedded as a normalized right transversal in a group with some universal property (see [14, Theorem 3.4, p.76]). Let  $\langle S \rangle$  be the subgroup of  $G$  generated by  $S$  and  $H_S$  be the subgroup  $\langle S \rangle \cap H$ . Then  $H_S = \langle \{xy(x \circ y)^{-1} | x, y \in S\} \rangle$  and  $H_S S = \langle S \rangle$  (see [14]).

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Identifying  $S$  with the set  $H \backslash G$  of all right cosets of  $H$  in  $G$ , we get a transitive permutation representation  $\chi_S : G \rightarrow \text{Sym}(S)$  defined by  $\{\chi_S(g)(x)\} = S \cap Hxg$ ,  $g \in G, x \in S$ . The kernel  $\ker \chi_S$  of this action is  $\text{Core}_G(H)$ , the core of  $H$  in  $G$ .

Let  $G_S = \chi_S(H_S)$ . This group is known as the *group torsion* of the right loop  $S$  (see [14, Definition 3.1, p.75]). The group  $G_S$  depends only on the right loop structure  $\circ$  on  $S$  and not on the subgroup  $H$ . Since  $\chi_S$  is injective on  $S$  and if we identify  $S$  with  $\chi_S(S)$ , then  $\chi_S(\langle S \rangle) = G_S S$  which also depends only on the right loop  $S$  and  $S$  is an NRT of  $G_S$  in  $G_S S$ . One can also verify that  $\ker(\chi_S|_{H_S S} : H_S S \rightarrow G_S S) = \ker(\chi_S|_{H_S} : H_S \rightarrow G_S) = \text{Core}_{H_S S}(H_S)$  and  $\chi_S|_S = \text{the identity map on } S$ . If  $H$  is a corefree subgroup of  $G$ , then there exists an NRT  $T$  of  $H$  in  $G$  which generates  $G$  (see [7]). In this case,  $G = H_T T \cong G_T T$  and  $H = H_T \cong G_T$ . Also  $(S, \circ)$  is a group if and only if  $G_S$  is trivial.

Let  $\mathcal{T}(G, H)$  denote the set of all normalized right transversals (NRTs) of  $H$  in  $G$ . Two NRTs  $S, T \in \mathcal{T}(G, H)$  are said to be *isomorphic* (denoted by  $S \cong T$ ), if their induced right loop structures are isomorphic. A subgroup  $H$  is normal in  $G$  if and only if all NRTs of  $H$  in  $G$  are isomorphic to  $G/H$  (see [14, p.70]).

Let  $V$  be a set. Denote by  $E(V) = \{\{u, v\} | u, v \in V, u \neq v\}$ , the 2-sets of  $V$ . A pair  $\Gamma = (V, E)$  with  $E \subseteq E(V)$  is called a graph on  $V$  (see [10]). The elements of  $V$  are called the *vertices* of  $\Gamma$  and those of  $E$  the edges of  $\Gamma$ . If  $\{u, v\} \in E$ , then we say that  $u$  and  $v$  are adjacent. The number  $|V|$  is called the order of  $\Gamma$ . A graph of order 0 or 1 is called as the trivial graph. The graph  $(\emptyset, \emptyset)$  is called as the empty graph. The graph  $\Gamma$  is called as *complete* if any two vertices are adjacent.

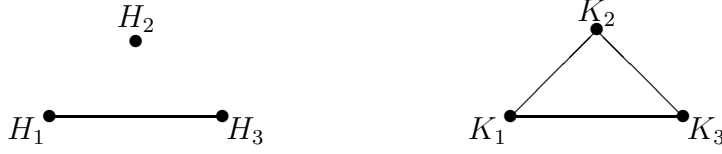
## 2 Transiso Graph

Let  $G$  be a finite group and  $d$  be divisor of  $|G|$  (order of  $G$ ). Let  $V_d$  be the set of all subgroups of  $G$  of order  $d$ . We define a graph  $\Gamma_d(G) = (V_d, E_d)$  with  $\{H_1, H_2\} \in E_d$  if and only if there exists  $S_i \in \mathcal{T}(G, H_i)$  ( $i = 1, 2$ ) such that  $S_1 \cong S_2$  with respect to the right loop structure induced on  $S_i$ . We will call this graph a *transiso graph*.

**Example 2.1.** 1. A finite cyclic group  $C_n$  of order  $n$  has unique subgroup corresponding to each divisor  $d$  of  $n$  so  $\Gamma_d(C_n)$  is pointed graph for each divisor  $d$  of  $n$ .

2. Let  $G = C_2 \times C_4 = \langle a, b; a^2, b^4, aba^{-1}b^{-1} \rangle$ . One can easily observe that  $V_2 = \{H_1 = \langle a \rangle, H_2 = \langle b^2 \rangle, H_3 = \langle ab^2 \rangle\}$  and  $V_4 = \{K_1 = \langle b \rangle, K_2 =$

$\langle ab \rangle, H_3 = \langle a, b^2 \rangle\}$ . Also  $H_1 \cong H_2 \cong H_3 \cong C_2$ ,  $G/H_1 \cong G/H_3 \cong C_4$ ,  $G/H_2 \cong C_2 \times C_2$ ,  $K_1 \cong K_2 \cong C_4$ ,  $K_3 \cong C_2 \times C_2$ ,  $G/K_1 \cong G/K_2 \cong G/K_3 \cong C_2$ . We show the connectivity of subgroups in following pictorial form:



**Proposition 2.2.** *A subgroup of a group  $G$  is always adjacent with its automorphic images in  $\Gamma_d(G)$  for any divisor  $d$  of  $|G|$ .*

*Proof.* Let  $H$  be a subgroup of a group  $G$ . Let  $f$  be an automorphism of  $G$  and  $K = f(H)$ . Choose  $S_1 \in \mathcal{T}(G, H)$ . Let  $S_2 = f(S_1)$ . Observe that  $S_2 \in \mathcal{T}(G, K)$ .

Let  $x, y \in S_1$ . Then  $\{f(x \circ y)\} = f\{x \circ y\} = f(S_1 \cap Hxy) = f(S_1) \cap f(Hxy) = S_2 \cap Kf(x)f(y) = \{f(x) \circ f(y)\}$ . This implies that  $f|_{S_1} : S_1 \rightarrow S_2$  is a right loop isomorphism. Hence,  $H$  and  $K$  are adjacent in  $\Gamma_d(G)$ .  $\square$

Converse of Proposition 2.2 is not true in general (see Example 2.1). Let  $\delta(G)$  denotes the number of divisors of  $|G|$  for which there is a subgroup of  $G$  of that order.

**Corollary 2.3.** *If number of orbits of the action of  $\text{Aut}(G)$  on the set  $V$  of all subgroups of  $G$  is equal to  $\delta(G)$ , then  $\Gamma_d(G)$  is complete for each divisor  $d$  of  $|G|$ .*

**Proposition 2.4.** *Let  $H_1$  and  $H_2$  be corefree subgroups of  $G$ . Let  $S_i \in \mathcal{T}(G, H_i)$  ( $i = 1, 2$ ) such that  $S_1 \cong S_2$  and  $\langle S_i \rangle = G$ . Then an isomorphism between  $S_1$  and  $S_2$  can be extended to an automorphism of  $G$  which sends  $H_1$  onto  $H_2$ .*

*Proof.* Note that  $H_{S_i} = \langle S_i \rangle \cap H_i = G \cap H_i = H_i$  ( $i = 1, 2$ ) and hence  $H_{S_i} S_i = G$ . Let  $p : S_1 \rightarrow S_2$  be a right loop isomorphism. It gives rise to an isomorphism  $\tilde{p} : G_{S_1} S_1 \rightarrow G_{S_2} S_2$  such that  $\tilde{p}(G_{S_1}) = G_{S_2}$  (see the discussion following [18, Lemma 2.5, p. 2684]). Since  $H_i$  ( $i = 1, 2$ ) is corefree,  $\chi_{S_i}$  defined in the section 1 is an isomorphism. Now, one can note  $\chi_{S_2}^{-1} \circ \tilde{p} \circ \chi_{S_1}$  is an automorphism of  $G$  which is an extension of  $p$  and sends  $H_1$  to  $H_2$ .  $\square$

Let  $H$  be a subgroup of  $\text{Sym}(n)$  of order 2 and  $S \in \mathcal{T}(\text{Sym}(n), H)$ . Then  $H_S = \langle S \rangle \cap H$  has order at most 2. Assume that  $|H_S| = 2$ . Then  $H_S = H$ , so  $|\langle S \rangle| = |HS| = n!$  and hence  $\langle S \rangle = \text{Sym}(n)$ . Assume that  $|H_S| = 1$ . Then  $S$  is a subgroup of  $\text{Sym}(n)$  of order  $\frac{n!}{2}$ . One can note that  $H$  can not be generated by an even permutation of order 2. In this case,  $S = \text{Alt}(n)$ , where  $\text{Alt}(m)$  denotes the alternating group of degree  $m$ .

- Example 2.5.** 1. Let  $H$  and  $K$  be distinct subgroups of  $\text{Sym}(n)$  of order 2 generated by odd permutations. Then one can observe that  $\text{Alt}(n) \in \mathcal{T}(\text{Sym}(n), H) \cap \mathcal{T}(\text{Sym}(n), K)$ . Therefore  $H$  and  $K$  are adjacent in  $\Gamma_2(\text{Sym}(n))$ .
2. Let  $H$  and  $K$  be subgroups of  $\text{Sym}(n)$  of order 2 generated by even permutations. Then as argued in the above paragraph, all NRTs of  $H$  and  $K$  generate  $\text{Sym}(n)$ . If  $H$  and  $K$  are adjacent, then by Proposition 2.4 there is an automorphism sending  $H$  onto  $K$ . If  $n \neq 6$ , then  $H$  and  $K$  conjugate (for  $\text{Aut}(\text{Sym}(n)) \cong \text{Inn}(\text{Sym}(n))$  for  $n \neq 6$  (see [19, p.300])). If  $n = 6$ , then  $H$  and  $K$  are again conjugate (for all automorphisms of  $\text{Sym}(6)$  send a permutation of cycle type  $(2, 2, 1, 1)$  to permutation of cycle type  $(2, 2, 1, 1)$ ). Moreover, if  $H$  and  $K$  are conjugate, then by Proposition 2.2  $H$  and  $K$  are adjacent.
3. Now if  $H$  is generated by an even permutation of order 2 and  $K$  is generated by an odd permutation of order 2, then they are not adjacent in  $\Gamma_2(\text{Sym}(n))$  (for otherwise they will be conjugate).
4. Since all subgroups of  $\text{Alt}(n)$  ( $n \leq 5$ ) of same order are conjugate, by Proposition 2.2  $\Gamma_d(\text{Alt}(n))$  ( $n \leq 5$ ) is complete for each divisor  $d$  of  $\frac{n!}{2}$ . As argued above, one can observe that  $\Gamma_2(\text{Alt}(n))$  ( $n \geq 8$ ) is not complete.
5. Let  $n \geq 6$ . Let  $H_1$  and  $H_2$  be two subgroups of  $\text{Alt}(n)$  such that  $H_1 \cong C_4$  and  $H_2 \cong C_2 \times C_2$ . Since  $\text{Alt}(m)$  ( $m \geq 5$ ) has no subgroup of index less than  $m$ ,  $|H_{S_i}| \neq 1$  and  $|H_{S_i}| \neq 2$  ( $i = 1, 2$ ). This implies that  $|H_{S_i}| = 4$  ( $i = 1, 2$ ). Hence, all members of  $\mathcal{T}(\text{Alt}(n), H_i)$  generates  $\text{Alt}(n)$ . Now, by Proposition 2.4 one observes that  $H_1$  and  $H_2$  are not adjacent in  $\Gamma_4(\text{Alt}(n))$ .

**Proposition 2.6.** For a finite abelian group  $G$ ,  $\Gamma_d(G)$  is complete for each divisor  $d$  of  $|G|$  if and only if each sylow subgroup of  $G$  is either elementary abelian or cyclic.

*Proof.* One can easily check the 'if' part. We will observe the 'only if' part.

Assume that  $G$  is not isomorphic to the group stated in the proposition and  $\Gamma_d(G)$  is complete for each divisor  $d$  of  $|G|$ . Then by Fundamental Theorem of abelian groups, there must be a summand of  $G$  isomorphic to  $C_{p^\alpha} \times C_{p^\beta}$  for some prime divisor  $p$  of  $|G|$ , where  $\alpha, \beta$  are positive integers and atleast one of  $\alpha$  and  $\beta$  is greater than 1. Without any loss, let us assume that  $\alpha > 1$  and  $C_{p^\alpha} \times C_{p^\beta}$  is at first place.

If  $\alpha > \beta \geq 1$ , then there exist two subgroups  $H_1$  and  $H_2$  such that  $H_1 \cong C_p \times \{1\} \times \cdots$  and  $H_2 \cong \{1\} \times C_p \times \cdots$ . One can observe that  $G/H_1 \not\cong G/H_2$ . This is a contradiction.

If  $\alpha = \beta > 1$ , then there exist two subgroups  $H_1$  and  $H_2$  such that  $H_1 \cong C_{p^2} \times \{1\} \times \cdots$  and  $H_2 \cong C_p \times C_p \times \cdots$ . Then  $G/H_1 \not\cong G/H_2$ . This is again a contradiction.  $\square$

**Corollary 2.7.** *If  $G$  is elementary abelian group, then  $\Gamma_d(G)$  is complete for each divisor  $d$  of  $|G|$ .*

Let  $D_{2n}$  denotes the dihedral group of order  $2n$ . We need following elementary lemma to prove that  $\Gamma_d(D_{2n})$  is complete for each divisor  $d$  of  $|G|$ . The proof of following Lemma can be found in [16, Theorem 2.37, p. 54] and [8, Theorem 3.3, p. 5].

**Lemma 2.8.** *A subgroup of a dihedral group  $D_{2n} = \langle a, b; a^n, b^2, (ba)^2 \rangle$  is either cyclic or dihedral. Moreover if  $m$  is a divisor of  $2n$  and*

1.  *$m$  is odd then all  $m$  subgroups of index  $m$  are conjugate to  $\langle a^m, b \rangle$ .*
2.  *$m$  is even and  $m$  does not divide  $n$  then there is only one subgroup  $\langle a^{\frac{m}{2}} \rangle$  of index  $m$ .*
3.  *$m$  is even and  $m$  divides  $n$  then a subgroup of index  $m$  is either  $\langle a^{\frac{m}{2}} \rangle$  or conjugate to exactly one of  $\langle a^m, b \rangle$  or  $\langle a^m, ba \rangle$ .*

**Proposition 2.9.** *Let  $D_{2n}$  denote the dihedral group of order  $2n$ . Then  $\Gamma_d(D_{2n})$  is complete for each divisor  $d$  of  $2n$ .*

*Proof.* Let  $G = D_{2n} = \langle a, b; a^n, b^2, (ba)^2 \rangle$ . Let  $d$  be a divisor of  $2n$  and  $m = \frac{2n}{d}$ .

Assume that  $m$  is odd. Then by Lemma 2.8, there are  $m$  subgroups of  $G$  of order  $d$  and all are conjugate to  $\langle a^m, b \rangle$ . Therefore, by Proposition 2.2,  $\Gamma_d(G)$  is complete for the divisor  $d$  for which  $m$  is odd.

Assume that  $m$  is even and  $m$  does not divide  $n$ . Then, by Lemma 2.8 there is only one subgroup  $\langle a^{\frac{m}{2}} \rangle$  of order  $d$ . Therefore,  $\Gamma_d(G)$  is complete for the divisor  $d$  for which  $m$  is even and  $m$  does not divide  $n$ .

Finally, assume that  $m$  is even and  $m$  divides  $n$ . Then by Lemma 2.8, a subgroup of order  $d$  is either  $\langle a^{\frac{m}{2}} \rangle$  or conjugate to exactly one of  $\langle a^m, b \rangle$  or  $\langle a^m, ba \rangle$ . Let  $H_1 = \langle a^{\frac{m}{2}} \rangle$ ,  $H_2 = \langle a^m, b \rangle$  and  $H_3 = \langle a^m, ba \rangle$ . Note that  $H_1$  is a normal subgroup of  $G$ . Hence, all NRTs of  $H_1$  in  $G$  are isomorphic to  $G/H_1 \cong D_{2 \cdot \frac{m}{2}}$ .

Now choose  $S_2 = \{1, a^2, a^4, \dots, a^{m-2}, ba, ba^3, ba^5, \dots, ba^{m-1}\} = \{a^{2i}, ba^{2j+1} | 0 \leq i, j \leq (\frac{m}{2} - 1)\}$  in  $\mathcal{T}(D_{2n}, H_2)$  and  $S_3 = \{a^{2i}, ba^{2j} | 0 \leq i, j \leq (\frac{m}{2} - 1)\}$  in  $\mathcal{T}(D_{2n}, H_3)$ . Note that  $\langle S_2 \rangle = \langle a^2, ba \rangle$  and  $\langle S_3 \rangle = \langle a^2, b \rangle$ . Then  $H_{S_2} = \langle S_2 \rangle \cap H_2 = \langle a^m \rangle \trianglelefteq \langle S_2 \rangle$  and  $H_{S_3} = \langle S_3 \rangle \cap H_3 = \langle a^m \rangle \trianglelefteq \langle S_3 \rangle$ . Therefore  $G_{S_2} = G_{S_3} = \{1\}$  and hence  $S_2$  and  $S_3$  are groups.

Let  $\circ_2$  denote the induced binary operation on  $S_2$  as described in the first paragraph of section 1. One can observe that,  $(a^2)^{\frac{m}{2}} = (ba)^2 = (ba \circ_2 a^2)^2 = 1$ . This implies that  $S_2 \cong D_{2 \cdot \frac{m}{2}}$ . Similarly  $S_3 \cong D_{2 \cdot \frac{m}{2}}$ . This shows that  $\Gamma_d(G)$  is complete also in this case.  $\square$

Let  $Q_8$  denote the quaternion group of order 8. Then each subgroup of  $Q_8$  is normal in  $Q_8$ . Hence,  $\Gamma_d(Q_8)$  is complete graph for each divisor  $d$  of 8. Note that each NRT of subgroups of order 2 generates  $Q_8$ . The converse of this is also true as observed below:

**Proposition 2.10.** *Let  $G$  be a non-abelian finite  $p$ -group with all NRTs of a subgroup of index greater than  $p$  generate  $G$  and  $\Gamma_d(G)$  is complete for each divisor  $d$  of  $|G|$ . Then  $G \cong Q_8$ .*

*Proof.* Let  $H_1$  and  $H_2$  be subgroups of  $G$  of same index greater than  $p$ . Since each finite  $p$ -group has a normal subgroup for each divisor of  $|G|$ , we can take  $H_2 \trianglelefteq G$ . Choose  $S_1 \in \mathcal{T}(G, H_1)$ ,  $S_2 \in \mathcal{T}(G, H_2)$  such that  $S_1 \cong S_2$ . By the assumption  $\langle S_1 \rangle = \langle S_2 \rangle = G$ . This implies that  $H_{S_1} = H_1$ ,  $H_{S_2} = H_2$ . Since  $H_2 \trianglelefteq G$ ,  $S_2$  is a group. This implies that  $G_{S_2} = \{1\}$ . Since  $S_1 \cong S_2$ ,  $G_{S_1} \cong G_{S_2}$  (see the discussion following [18, Lemma 2.5, p. 2684]). Since  $G_{S_1} \cong H_{S_1}/\text{Core}_G H_{S_1} = H_1/\text{Core}_G(H_1)$ ,  $H_1$  is also normal in  $G$ . Hence  $G$  is a Dedekind group (see [15, p.143]). Since  $G$  is finite  $p$ -group, by [15, Theorem 5.3.7, p. 143]  $G$  is isomorphic to  $Q_8$  or  $Q_8 \times A$ , where  $A$  is an elementary abelian 2-group. Assume that  $G \cong Q_8 \times A$ . Let  $H$  be a subgroup of  $A$  of order 2. Then one can note that an NRT of  $H$  in  $G$  does not generate  $G$ . This is a contradiction. Thus  $G \cong Q_8$ .  $\square$

### 3 Complete Transiso Graph For Lower Prime Power Order Groups

In this section, we determine the completeness of the transiso graph for finite  $p$ -group  $G$  for divisor  $p$  ( $p$  an odd prime) upto order  $p^5$ . We will show that if  $|G| = p^4$ , then transiso graph  $\Gamma_p(G)$  is not complete. For the group of order  $p^5$ ,  $\Gamma_p(G)$  is not complete except  $\Phi(G) = G' = Z(G) \cong C_p \times C_p$ , where  $\Phi(G)$ ,  $G'$ ,  $Z(G)$  and  $C_p$  denotes the Frattini subgroup, commutator subgroup, center of  $G$  and cyclic group of order  $p$  respectively. Using the small group library of GAP ([11]), we found that the transiso graph  $\Gamma_3(G)$  for the  $37^{th}$  group of order  $3^5$  is complete. This group is of exponent 3. We will observe that for the extra special group  $G$  of order  $p^3$  of exponent  $p$ , the transiso graph  $\Gamma_d(G)$  is complete for each divisor  $d$  of  $p^3$ .

We further ask to determine the structure of the group  $G$  of order  $p^n$  for which transiso graph  $\Gamma_d(G)$  is complete for each divisor  $d$  of  $p^n$ . This problem can be thought as a dual problem posed in [1]. In [1], R. Armstrong proves only finite non-abelian  $p$ -group all of whose subgroups of same order are isomorphic is the group of order  $p^3$  of exponent  $p$ . In case of a complete transiso graph we feel that the only finite non-abelian  $p$ -group for which transiso graph  $\Gamma_d(G)$  is complete for each order  $d$  is the group of order  $p^3$  of exponent  $p$ .

Throughout the section, we will adopt following convention. The prime  $p$  will always be odd. The group  $G$  denotes finite  $p$ -group which is not  $p$ -central, that is  $G$  has non-central subgroup of order  $p$ . Whenever we write  $H$  or  $H_i$  ( $i \in \mathbb{N}$ ), we will always mean that this is a non-normal subgroup of  $G$  of order  $p$ . Whenever we write the semidirect product  $G_1 \rtimes G_2$  of the groups  $G_1$  and  $G_2$ , we will mean that it is not a direct product.

**Proposition 3.1.** *Let  $G$  be a non  $p$ -central finite  $p$ -group. Then  $\Gamma_p(G)$  is complete if and only if whenever  $H$  is a non-normal subgroup of  $G$  of order  $p$ ,  $G \cong H \rtimes K$  for some subgroup  $K$  of  $G$  with  $G/L \cong K$  for any normal subgroup  $L$  of  $G$  of order  $p$ .*

*Proof.* One can easily observe the 'if' part. We will only prove 'only if' part. Let  $H \not\trianglelefteq G$  of order  $p$ . Let  $L$  be any normal subgroup of  $G$  of order  $p$ . Since  $\Gamma_p(G)$  is complete, there exists  $S_1 \in \mathcal{T}(G, H)$  and  $S_2 \in \mathcal{T}(G, L)$  such that  $S_1 \cong S_2$ . This implies that  $G_{S_1} \cong G_{S_2}$  (see the discussion following [18, Lemma 2.5, p. 2684]). Since  $L \trianglelefteq G$ ,  $G_{S_2} = \{1\}$ . Also, since  $H$  is core-free subgroup of  $G$ ,  $H_{S_1} \cong G_{S_1}$ . This implies that  $H_{S_1} \cong G_{S_1} \cong G_{S_2} = \{1\}$ . This means that  $S_1$  is subgroup of  $G$ . Denote it by  $K$ . Note that  $K \trianglelefteq G$ . This implies that  $G \cong H \rtimes K$  and  $K \cong G/L$  for any  $L \trianglelefteq G$ .  $\square$

**Corollary 3.2.** *Let  $G$  be a non  $p$ -central finite  $p$ -group with  $|\Phi(G)| = p$  and  $\Gamma_p(G)$  is complete. Then  $G \cong H \rtimes K$ , where  $K \cong C_p \times \cdots \times C_p$  ( $p-1$  times).*

**Proposition 3.3.** *Let  $G$  be a finite  $p$ -group ( $p$  odd prime) with the property that whenever  $H$  is a non-normal subgroup of  $G$  of order  $p$ ,  $G$  is the semidirect product of  $H$  and a normal subgroup  $K$  such that all subgroups of  $K$  of order  $p$  are normal in  $G$  and  $K$  is isomorphic to the quotient  $G/L$  for any normal subgroup  $L$  of  $G$  of order  $p$ . Then  $K$  is a cyclic group.*

*Proof.* Let  $H = \langle h \rangle$  be a non-normal subgroup of order  $p$  and  $G = H \rtimes K$ . Since  $K = G/L$  for any subgroup  $L$  of  $K$  of order  $p$  and all subgroups of  $K$  of order  $p$  are normal in  $G$ , it follows that the image  $HL/L$  of  $H$  in  $G/L$  is normal and hence also central in  $G/L$ . So  $[h, g] \in L$  for all  $g \in G$ . Now if  $K$  had another subgroup  $L'$  of order  $p$ , then we would also have  $[h, g] \in L'$  and hence  $[h, g] = 1$  so  $H \leq Z(G)$ , contrary to assumption. So  $K$  has a unique subgroup  $L$  of order  $p$ . This implies that  $K$  is a cyclic group.  $\square$

**Example 3.4.** One can note that  $\Gamma_p(G)$  is complete, if  $|G| \leq p^2$ . Let  $G$  be a group of order  $p^3$  and  $G$  be non-abelian. From the classification of group of order  $p^3$ , we note that  $|Z(G)| = |\Phi(G)| = p$ . By the classification of groups of order  $p^3$ , there are two non-abelian groups upto isomorphism. One is of exponent  $p$  and other is of exponent  $p^2$ . Let  $G$  be a non-abelian group of order  $p^3$  of exponent  $p^2$ . Assume that  $\Gamma_p(G)$  is complete. By Corollary 3.2,  $G \cong C_p \rtimes (C_p \times C_p)$ . One can note that there is unique subgroup  $K$  of  $G$  isomorphic to  $C_p \times C_p$ . Let  $H$  be non-normal subgroup of  $G$  of order  $p$  contained in  $K$ . Then  $G \not\cong H \rtimes K$ . Thus, by Proposition 3.1  $\Gamma_p(G)$  is not complete. This is a contradiction. Now, assume that  $G$  is non-abelian group of order  $p^3$  of exponent  $p$ . We will now prove that  $\Gamma_d(G)$  is complete for each divisor  $d$  of  $p^3$ .

First note that all subgroup of  $G$  of order  $p^2$  are normal in  $G$  and their quotient are isomorphic to  $C_p$ . Hence  $\Gamma_{p^2}(G)$  is complete.

Let  $H$  be any non normal subgroup of order  $p$ . Choose  $x \in G \setminus (Z(G)H)$ . Take  $K = \langle x \rangle Z(G)$ . One can note that  $G \cong H \rtimes K$ . By Proposition 3.1,  $\Gamma_p(G)$  is complete.

**Example 3.5.** In Example 3.4, we have seen that if  $G$  is of order  $p^3$  of exponent  $p$ , then  $\Gamma_d(G)$  is complete for each divisor  $d$  of  $p^3$ . We now calculate number of vertices of trasiso graph. Note that vertices are subgroups of same order.

Since  $G$  is of exponent  $p$ , number of elements of order  $p$  is  $p^3 - 1$ . Also if  $H$  is a subgroup of  $G$  of order  $p$ , then there are  $p - 1$  elements in  $H$  of order  $p$ . Thus, there are  $\frac{p^3-1}{p-1} = p^2 + p + 1$  subgroups of order  $p$ .

Note that each subgroup of  $G$  of order  $p^2$  is isomorphic to  $C_p \times C_p$ . Let  $x$  and  $y$  be two elements of  $G$  of order  $p$  which generates a subgroup of order  $p^2$ . Note that  $xy = yx$ . This means that the order of the centralizer  $C_G(x)$  of  $x$  is atleast  $p^2$ . If  $|C_G(x)| = p^3$ , then  $x \in Z(G)$ . If  $|C_G(x)| = p^2$ , then  $C_G(x) = \langle x, y \rangle$ . This means that  $Z(G) \subseteq \langle x, y \rangle$ . Without any loss, we assume that  $x \in Z(G)$ . Therefore, to count subgroups of order  $p^2$  we only take care of choices for  $\langle y \rangle$  which are non central. There are  $p^2 + p$  choices for  $\langle y \rangle$ . Note that there are  $p$  subgroups of  $\langle x, y \rangle$  which are not central. Thus there are  $\frac{p^2+p}{p} = p + 1$  subgroups of order  $p^2$ .

Let  $G$  be a non-abelian group of order  $p^4$ . Then one can easily observe that  $\Phi(G)$  is abelian. Let  $G$  be a non-abelain group of order  $p^5$ . Then from a result of [4], which states that if  $G$  is a finite  $p$ -group ( $p$  odd prime) and the center  $Z(\Phi(G))$  is cyclic, then  $\Phi(G)$  is cyclic, one can observe that  $\Phi(G)$  is abelian in this case also. Now, we have following:

**Lemma 3.6.** Let  $G$  be a non-abelian finite  $p$ -group of order either  $p^4$  or  $p^5$ . Then  $\Phi(G)$  is not a cyclic group unless  $|\Phi(G)| = p$ .



*Proof.* Let  $|G| = p^4$ . Assume that  $\Phi(G) \cong C_{p^2}$ . By [5, Lemma 3.1, p. 304],  $G \cong Q \times E$ , where  $E$  is elementary abelian and both  $\Phi(Q)$  and  $Z(Q)$  are cyclic. Assume that  $|E| = p^2$ . Then  $|Q| = p^2$ . Hence  $G$  is abelian. This is a contradiction. Similarly  $|E| = p^3$  is not possible. Thus,  $|E| = p$ ,  $|Q| = p^3$  and  $Q$  will be a non abelian group. Since  $|Q| = p^3$ ,  $|\Phi(Q)| = p$ . But, by [13, Theorem, p. 22],  $\Phi(G) \cong \Phi(Q)$ . This is a contradiction.

Let  $|G| = p^5$ . If  $\Phi(G)$  is cyclic group of order greater than  $p$ , either  $\Phi(G) \cong C_{p^2}$  or  $\Phi(G) \cong C_{p^3}$ . Assume that  $\Phi(G) = C_{p^2}$ . As argued above, the cases  $|E| = p^4$  and  $|E| = p^3$  are not possible. Therefore, assume that  $|E| = p^2$ . Then  $|Q| = p^3$  and  $Q$  is non-abelian. Since  $|\Phi(Q)| = p$  and  $|\Phi(G)| = p^2$ , by [13, Theorem, p. 22] we get a contradiction. Assume that  $|E| = p$ . Then  $|Q| = p^4$ . By [13, Theorem, p. 22],  $\Phi(G) \cong \Phi(Q) \cong C_{p^2}$ . Since  $|Q| = p^4$ , we get a contradiction. Thus  $\Phi(G) \not\cong C_{p^2}$ . Similarly  $\Phi(G) \not\cong C_{p^3}$ .  $\square$

**Lemma 3.7.** *Let  $G$  be a finite  $p$ -group and  $\Gamma_p(G)$  is complete. Then all subgroups of order  $p$  which are contained in  $\Phi(G)$  are normal in  $G$ .*

*Proof.* Assume that  $H$  be subgroup of  $G$  contained in  $\Phi(G)$  which is not normal in  $G$ . Then by Proposition 3.1,  $G \cong H \rtimes K$  for some subgroup  $K$  of  $G$ . But by [9, Lemma 2.11, p. 1916],  $H \trianglelefteq G$ . This is a contradiction.  $\square$

**Corollary 3.8.** *Let  $G$  be a non-abelian group of order either  $p^4$  or  $p^5$ . Let  $|Z(G)| = p$  and  $\Gamma_p(G)$  is complete. Then  $\Phi(G) = Z(G)$ .*

**Proposition 3.9.** *Let  $G$  be a finite  $p$ -group such that the order of the commutator subgroup  $G'$  is  $p$ . If  $G' \subsetneq \Phi(G)$  and  $\Phi(G)$  is not cyclic, then  $\Gamma_p(G)$  is not complete.*

*Proof.* Assume that  $\Gamma_p(G)$  is complete. By Lemma 3.7, all subgroups of  $\Phi(G)$  of order  $p$  are normal in  $G$ . By Proposition 3.1,  $G \cong H \rtimes K$  and  $K \cong G/L$  for any normal subgroup  $L$  of order  $p$ . Take  $L_1 = G'$  and  $L_2 \subseteq \Phi(G)$  such that  $L_2 \neq L_1$  and  $|L_2| = p$ . By Proposition 3.1,  $G/L_1 \cong G/L_2$ . This implies that  $G' \subseteq L_2$ . This is a contradiction.  $\square$

**Corollary 3.10.** *Let  $G$  be a group of order  $p^4$  such that  $\Phi(G) = Z(G) \cong C_p \times C_p$ . Then  $\Gamma_p(G)$  is not complete.*

*Proof.* Since  $|\Phi(G)| = p^2$ ,  $G = \langle x, y \rangle$  for some  $x, y \in G$ . Since  $G' \leq Z(G)$ ,  $G' = \langle [x, y] \rangle$ . But since  $Z(G)$  is an elementary abelian group,  $|G'| = p$ . By Proposition 3.9,  $\Gamma_p(G)$  is not complete.  $\square$

We have following theorem for groups of order  $p^4$ :

**Theorem 3.11.** *Let  $G$  be a group of order  $p^4$ . Then  $\Gamma_p(G)$  is not complete.*

*Proof.* Assume that  $\Gamma_p(G)$  is complete. By Lemmas 3.6, 3.7 and Proposition 3.9, we are left with  $|G'| = |\Phi(G)| = p$ . Let  $H$  be a non-normal subgroup of  $G$ . By Proposition 3.1,  $G = H \rtimes K$  and  $K \cong G/\Phi(G) \cong C_p \times C_p \times C_p$ . By [3, §3, p. 64], we have only one choice for  $G$  whose numbering in above cited reference is given by 1. By [3, §3, p. 64],  $G' \cong C_p$  and  $Z(G) \cong C_p \times C_p$ . Since  $|G'| = p$ , we can choose a subgroup  $K$  of  $G$  of order  $p$  distinct from  $G'$ . By Proposition 3.1,  $G/K \cong G/G'$ . This is a contradiction.  $\square$

**Proposition 3.12.** *Let  $G$  be a group of order  $p^5$  such that  $|\Phi(G)| = p$ . Then  $\Gamma_p(G)$  is not complete.*

*Proof.* Assume that  $\Gamma_p(G)$  is complete. Note that  $G' = \Phi(G)$ . By Corollary 3.2,  $G \cong H \rtimes K$  where  $H$  is a non-normal subgroup of  $G$  of order  $p$  and  $K \cong C_p \times C_p \times C_p \times C_p$ . By [3, §4, p. 65], we have only one choice for  $G$  which is numbered by 1 in this reference. By [3, §4, p. 65],  $Z(G) \cong C_p \times C_p \times C_p$ . Choose  $y \in Z(G) \setminus G'$ . Take  $L_1 = G'$  and  $L_2 = \langle y \rangle$ . Then one can observe that  $G/L_1 \not\cong G/L_2$ . This is a contradiction to the Proposition 3.1.  $\square$

**Proposition 3.13.** *Let  $G$  be group of order  $p^5$  such that  $|\Phi(G)| = p^3$ . Then  $\Gamma_p(G)$  is not complete.*

*Proof.* Assume that  $\Gamma_p(G)$  is complete. By Lemma 3.6,  $\Phi(G) \cong C_p \times C_p \times C_p$  or  $\Phi(G) \cong C_p \times C_{p^2}$ . Assume that  $\Phi(G) \cong C_p \times C_p \times C_p$ . By Lemma 3.7,  $\Phi(G) \subseteq Z(G)$ . Since  $|G| = p^5$ ,  $\Phi(G) = Z(G)$ . By [3], there is no group of order  $p^5$  such that  $|Z(G)| = p^3$  and  $|G'| = p^2$  and  $|Z(G)| = p^3$  and  $|G'| = p^3$ . Thus  $|G'| = p$ . But by Proposition 3.9, we get a contradiction. Thus  $\Phi(G) \cong C_p \times C_{p^2}$ .

By Lemma 3.7,  $Z(G)$  contains a subgroup isomorphic to  $C_p \times C_p$ . By [3], there is no group of order  $p^5$  such that  $|Z(G)| = p^3$  and  $G' \cong C_p \times C_{p^2}$  or  $|Z(G)| = p^3$  and  $|G'| = p^2$ . Hence  $Z(G) \cong C_p \times C_p$  and  $|G'|$  is either  $p$  or  $p^2$ . By Proposition 3.9,  $|G'| \neq p$ . Hence  $|G'| = p^2$ . Assume that  $G' \cong C_{p^2}$ . By [12, §4, p. 618], there is a group in isoclinism family  $\phi_8$ . But in this case,  $|Z(G)| = p$ . This is a contradiction. Thus  $G' \cong C_p \times C_p$ . By Lemma 3.7,  $Z(G) = G'$ . By [12, §4, p. 618], there are groups in isoclinism family  $\phi_4$ . But, in this case  $G/Z(G)$  is an elementary abelian  $p$ -group. This is a contradiction.  $\square$

**Proposition 3.14.** *Let  $G$  be group of order  $p^5$  such that  $|\Phi(G)| = p^2$ . Then  $\Gamma_p(G)$  is not complete unless  $\Phi(G) = G' = Z(G) \cong C_p \times C_p$ .*

*Proof.* Let  $G$  be a group of order  $p^5$  other than the case  $\Phi(G) = G' = Z(G) \cong C_p \times C_p$ . Assume that  $\Gamma_p(G)$  is complete. By Lemma 3.6,  $\Phi(G) \cong C_p \times C_p$ . By Proposition 3.9,  $|G'| \neq p$ . Hence  $\Phi(G) = G'$ . By Lemma 3.7,  $\Phi(G) \subseteq Z(G)$ . By [3], there is no group of order  $p^5$  such that  $|Z(G)| = p^3$  and  $|G'| = p^2$ . Thus  $\Phi(G) = G' = Z(G) \cong C_p \times C_p$ . This is a contradiction.  $\square$

**Remark 3.15.** We write  $\text{SmallGroup}(n,m)$  for the  $m^{\text{th}}$  group of order  $n$  as quoted in the "Small Groups" library in GAP ([11]). Using GAP calculations, we found that  $\text{SmallGroup}(243,37)$  satisfies the condition of Proposition 3.1. Hence  $\Gamma_3(G)$  is complete, where  $G = \text{SmallGroup}(243,37)$ . One can check that  $G \cong C_3 \times (C_3 \times C_3 \times C_3 \times C_3)$  and  $\Phi(G) = G' = Z(G) \cong C_3 \times C_3$ .

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